# Solidification and Melting of Materials Subject to Convection and Radiation

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**One-dimensional heat transfer in a finite region of solid or liquid with phase change associated with radiative and convective boundary conditions at the fixed boundary surface is solved using both Biot's variational and Goodman's integral methods. The total solidification time and the rates of solidification obtained based on the present approximate analyses agree very well with the numerical solutions of an earlier work.**

## Nomenclature

$b$ = constant in Eq. (15)
$B$ = $L/C_pT_i$ for solidification
= $L/C_pT_e$ for melting
$B_o$ = $h\mathcal{L}/k$, Biot Number
$C$ = constant in Eq. (15)
$c$ = $\rho C_p$ heat capacity per unit volume
$c_p$ = specific heat
$F$ = overall radiation shape factor
$k$ = thermal conductivity
$H$ = heat flow vector
$\dot{H}$ = heat flux
$H$ = $\sigma FT_i^3\mathcal{L}/k$, radiation parameter for solidification
= $\sigma FT_e^3\mathcal{L}/k$, radiation parameter for melting
$h$ = coefficient of heat transfer
$L$ = latent heat
$\mathcal{L}$ = thickness of the slab
$p$ = 1 for solidification
= −1 for melting
$q_1$ = surface temperature at $x=0$
$S$ = $\delta/\mathcal{L}$, dimensionless solidification or melting distance
$\dot{S}$ = $dS/d\tau$, dimensionless propogation rate; i.e., the rate of solidification for liquid or the rate of melting for solid
$t$ = time
$T$ = absolute temperature in solid or liquid
$T_i$ = initial temperature
$T_a$ = ambient temperature
$T_e$ = environment temperature
$U$ = dimensionless temperature
= $T/T_i$ for solidification
= $T/T_e$ for melting
$U_a$ = dimensionless ambient temperature
= $T_a/T_i$ for solidification
= $T_a/T_e$ for melting
$U_e$ = dimensionless environment temperature
= $T_e/T_i$ for solidification
= 1 for melting
$U_i$ = dimensionless initial temperature
= 1 for solidification
= $T_i/T_e$ for melting
$U_o$ = dimensionless surface temperature
= $q_1/T_i$ for solidification
= $q_1/T_e$ for melting
$x$ = dimensional space coordinate
$X$ = dimensionless space coordinate, $x/\mathcal{L}$
$Z$ = $B_oS$
$\rho$ = density
$\theta$ = temperature in solid or liquid above the initial temperature, $T-T_i$
$\sigma$ = Stefan-Boltzmann constant
$\alpha$ = thermal diffusivity, $k/\rho C_p$
$\tau$ = dimensionless time, $\alpha t/\mathcal{L}^2$
$\delta$ = solidification or melting line
$\tau'$ = $B_o^2\tau$

Received August 14, 1974; revision received February 3, 1975.
Index categories: Heat Conduction; Radiation and Radiative Heat Transfer.

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## Introduction

PROBLEMS of transient heat conduction involving phase change are of practical importance. These problems occur frequently during the aerodynamic heating of space vehicles, nuclear reactor operation, soil consolidation, casting of metals in molds, heat exchangers using cryogenics, and freezing ice on rivers. The heat conduction with phase change differs from the heat conduction without phase change in that the interface between the solid and liquid phases is moving and hence the boundary condition at this interface is nonlinear. Therefore, exact solutions are only limited to a few simple cases.[1,2] As a consequence, approximate methods of analytical solution become very important and practical. Two approximate techniques are found to be useful in the solution of heat conduction with moving boundary. They are the variational method due to Biot[3-5] and heat balance integral method due to Goodman.[6] These techniques have been successfully applied by many investigators[7-12] to phase change problems with: 1) constant heat flux, 2) constant surface temperature, and 3) convective boundary conditions. However, none of the analyses previously mentioned have included the radiative boundary condition, a case of practical significance such as metal casting at high temperatures.

The purpose of this work is to extend the applicability of Biot's and Goodman's methods to the problems of solidification and melting caused by aerodynamic and radiative cooling or heating. In particular, the temperature distribution and the location of interface (melt-line or solidification-line) are to be determined. As will be seen later, the total solidification time predicted based on the present analysis agrees very well with those of Goodling and Khader[13,14] obtained directly using a finite difference scheme.

## Formulation of the Problem

Consideration is given to a finite region of liquid (or solid) which is initially at its fusion (or melting) temperature, $T_i$. Because of radiative and aerodynamic cooling (or heating), phase change takes place. Let $T_e$ and $T_a$ be the environment and ambient temperatures, respectively, and $t$ be the time measured from the initiation of phase change. Assuming that the thermal properties are constant and the newly formed

solid (or liquid) is opaque to thermal radiation, we can write the energy equation describing the system in the following form:

$$\partial T/\partial t=\alpha(\partial^2 T/\partial x^2)\, t>0,\ 0\le x\le\delta(t) \tag{1}$$

$$T=T_i\ @\ t=0 \tag{2}$$

$$\partial T/\partial x=0\ @\ x=\mathcal{L} \tag{3}$$

$$k(\partial T/\partial x)=\sigma F(T^4-T_e{}^4) +h(T-T_a)@\ x=0 \tag{4}$$

$$k(\partial T/\partial x)=p\,\rho L\,\mathrm{d}\delta/\mathrm{d}t\ @\ x=\delta(t) \tag{5}$$

$$T=T_i\ @\ x=\delta(t) \tag{6}$$

where $\delta$ is the location of interface between the solid and liquid, $p=1$ represents the case of solidification; $p=-1$, for melting. Notice that Eq. (3) implies the finiteness of the liquid or solid region and guarantees the condition $T=T_i$ as $x>\delta(t)$. $F$ is overall radiation shape factor which is assumed to be constant for simplicity. Analysis involving temperature dependent shape factor but without phase change is available elsewhere.[15]

## Approximate Solutions

### Variational Method

Using Biot's variational principle, we first define a thermal flow vector $\boldsymbol{H}$ whose time rate of change, $\dot{\boldsymbol{H}}$ is the heat flux across an area normal to $\boldsymbol{H}$. The energy balance represented by Eq. (1) may be in terms of $\boldsymbol{H}$ as

$$\mathrm{div}\,\boldsymbol{H}=-c\theta \tag{7}$$

where $c$ is heat capacity and $\theta$ is the temperature above the initial temperature. We then assume a simple linear temperature profile inside the newly formed phase in terms of surface temperature $q_1$ and the location of phase boundary, $\delta$

$$\theta=(q_1-T_i)(1-x/\delta) \tag{8}$$

Equations (7) and (8) give rise to the expressions of heat flow vector and the heat flux of the forms

$$H=1/2\ c\delta(q_1-T_i)(1-x/\delta)^2-p\rho L\delta \tag{9}$$

$$\dot{H}=1/2\ c\dot{\delta}(q_1-T_i)(1-x/\delta)^2+1/2\ c\delta\dot{q}_1(1-x/\delta)^2 +c(q_1-T_i)(1-x/\delta)x\dot{\delta}/\delta-p\rho L\dot{\delta} \tag{10}$$

Once the temperature distribution and the heat flowfield are known, Biot's variational principle can be applied which leads to the following expression for the geometry under consideration

$$\int_o^\delta (c\theta\frac{\partial\theta}{\partial\delta}+\frac{\dot{H}}{k}\frac{\partial\dot{H}}{\partial\dot{\delta}}\mathrm{d}x=(\theta\frac{\partial H}{\partial\delta})_{x=0} \tag{11}$$

The right hand side of the above equation represents the "generalized thermal force"[5] at the surface. Substituting Eqs. (8-10) into Eq. (11) gives

$$B^2S\,\mathrm{d}S/\mathrm{d}\tau-2/3\ pB(U_o-U_i)S\,\mathrm{d}S/\mathrm{d}\tau +2/15(U_o-U_i)^2S\,\mathrm{d}S/\mathrm{d}\tau +3/40\ S^2(U_o-U_i)\mathrm{d}U_o/\mathrm{d}\tau -1/6\ pBS^2(\mathrm{d}U_o/\mathrm{d}\tau)=1/3(U_o-U_i)^2-pB(U_o-U_i) \tag{12}$$

where $S=\delta/\mathcal{L}$, $\tau=\alpha t/\mathcal{L}^2$, $B_o=h\mathcal{L}/k$ and $B=L/c_pT_i$ or $L/c_pT_e$, $U_i=1$ or $T_i/T_e$, $U_o=q_1/T_i$ or $q_1/T_e$ depending solidification or melting. Equation (12) contains two unknowns, namely $S$ and $U_o$. The combination of Eqs. (4) and (8) provides a relationship between these two variables:

$$S=(U_i-U_o)/F(U_o) \tag{13}$$

where

$$F(U_o)=H(U^4{}_o-U_e{}^4)+B_o(U_o-U_a)$$

$$H=\sigma FT_i{}^3\mathcal{L}/k \text{ or, } \sigma FT_e{}^3\mathcal{L}/k,$$

$$U_a=(T_a/T_i) \text{ or, } (T_a/T_e), \text{ and}$$

$$U_e=(T_e/T_i) \text{ or } 1$$

Eliminating $S$ from Eqs. (12) and (13) yields an expression for the surface temperature of the form

$$\mathrm{d}U_o/\mathrm{d}\tau=f_1(U_o) \tag{14}$$

$$f_1(U_o)=F^3(U_o)(U_o-U_i-3pB)/[3B^2-2pB(U_o-U_i)+2/15(U_o-U_i)^2]\ [F(U_o-(U_o-U_i)(4HU_o{}^3+B_o)]+(U_o-U_i)F(U_o)[9/40(U_o-U_i)-BP/2] \tag{14a}$$

With initial condition $U_o(0)=U_i$, Eq. (14) can be easily integrated numerically. Once the surface temperature is obtained, the fusion (or melting) line and the temperature distribution can be determined from Eqs. (13) and (8), respectively.

### Heat Balance Integral Method

We assume a dimensionless temperature profile inside the newly formed solid (or liquid) of the form

$$U=U_i+b(X-S)+C(X-S)^2 \tag{15}$$

Equations (1) and (5) provides a useful relation at liquid-solid boundary, i.e.,

$$1/pB(\partial U/\partial X)^2+\partial^2U/\partial X^2=0@X=S(\tau) \tag{16}$$

The coefficients in Eq. (15) can be determined from Eqs. (4) and (16). They are

$$b=[-1+(1+4SF(U_o)/pB)^{1/2}]pB/2S \tag{17}$$

$$C=(1/2S)[b-F(U_o)] \tag{18}$$

Integrating Eq. (1) from $X=0$ to $X=S\tau$ and making use of Eqs. (2, 4, and 16) leads to the following expression

$$-pB\,\mathrm{d}S/\mathrm{d}\tau+F(U_o)=bS\,\mathrm{d}S/\mathrm{d}\tau-CS^2\,\mathrm{d}S/\mathrm{d}\tau +(1/2)\ S^2\,\mathrm{d}b/\mathrm{d}\tau-(1/3)\ S^3\,\mathrm{d}C/\mathrm{d}\tau \tag{19}$$

Setting $X$ equal to zero in Eq. (15) and differentiating the result with respect to time yields

$$\mathrm{d}U_o/\mathrm{d}\tau=-b\,\mathrm{d}S/\mathrm{d}\tau-S\,\mathrm{d}b/\mathrm{d}\tau +2CS\,\mathrm{d}S/\mathrm{d}\tau+S^2\,\mathrm{d}C/\mathrm{d}\tau \tag{20}$$

With the aid of Eqs. (17) and (18), the coefficients $b$ and $C$ in Eqs. (19) and (20) can be eliminated. Consequently, we obtain the following two simultaneous ordinary differential equations

$$\dot{S}=\mathrm{d}S/\mathrm{d}\tau=f_2(U_o, S) \tag{21}$$

$$\dot{U}_o=\mathrm{d}U_o/\mathrm{d}\tau=f_3(U_o, S) \tag{22}$$

along with the initial conditions $U_o(0)=1$ and $S=(0)$, where

$$f_2(U_o, S)=[F(U_o)/g_1(U_o, S)] \quad (21a)$$
$$+1/2]\ [SF(U_o)/g_4(U_o, S)+SF(U_o)]\}$$

$$\div[2+\frac{S(4HU_o^3+B_o)}{g_4(U_o, S)}+S(4HU_o^3+B_o)] \quad (21b)$$

$$f_3(U_o, S)=f_2(U_o, S)g_2(U_o,S)/g_3(U_o, S) \quad (22a)$$

$$g_1(U_o, S)=5pB/6+pBg_4(U_o, S)/3$$
$$-\frac{pB+2SF(U_o)}{6g_4(U_o, S)})+SF(U_o)/3$$
$$-\{(1/3)S(4HU_o^3+B_o)[1/g_4(U_o, S)$$

$$g_2(U_o, S)=-1/2\ F(U_o)[1/g_4(U_o, S)+1] \quad (22b)$$

$$g_3(U_o,S)=1+(1/2)\,S(4HU_o^3+B_o)$$
$$\times[1+1/g_4(U_o,S)] \quad (22c)$$

$$g_4(U_o, S)=[(1+4SF(U_o)/pB)]^{1/2}$$

Therefore the complicated nonlinear problem of phase change is reduced to a simple initial value problem represented by Eqs. (21) and (22) which can be solved easily by using the standard techniques. In this work, a fourth order Runge-Kutta method, of which the program is available in IBM scientific sub-routine package is employed to evaluate $U_o$ and $S$. It should be noted that the computations of $U_o$ and $S$ based on Eqs. (13, 14, 21, and 22) are much simpler and straightforward as compared with the finite difference solution of Eqs. (1-6).

## Results and Discussion

Numerical results based on the present analysis are presented graphically. Only the case of solidification are illustrated here. Figures 1 and 2 show the time variant fusion front motion and the surface temperature history, respectively. Both Biot's number and radiation parameter are taken as unity and both temperature ratios, $T_e/T_i$ and $T_a/T_i$ are set equal to 0.25. The parameter $B$, which is essentially the inverse of Stefan number, ranges from 0.1-10. In all figures the solid lines denote the present results based on the variational method while the broken lines represent those of heat balance integral method. For comparison purposes, the finite difference solutions for the solidification rate presented in Ref. 14 are included in Fig. 1 and are shown by dash lines. As can be seen their numerical and the present approximate solutions are in remarkable agreement. More interestingly, the two approximate solutions, although appearing in completely different formats, agree very well with each other, especially for large values of parameter $B$.

The melting rate or solidification rate given by Eq. (21) [or from the differentiation of Eq. (13)] can be eaeily seen from the slope of the curves in Fig. 1. The total solidification time (time required for all liquid to become solid) corresponds to the value of $\tau$ when $S$ reaches unity, and can also be found from Fig. 1. Comparisons show that both solidification rate and the total solidification time predicted from the present analysis agree well with those of Ref. 14.

Using special transformations, Lin[16] presented an approximate analytical technique for the solidification problem with the convective boundary condition at one surface. Figure 3 shows comparisons between his results and the present analysis. The parameters $B$ and $B_o$ are arbitrarily set equal to 1 and 5, respectively. ($H=0$, since no radiation). The computations are based on $T_a/T_i=0.9$ and 0.75. (This corresponds to $\theta_c=0.1$ and 0.25, respectively, in Ref. 16.) It is seen that the agreement among the three approximate solutions is excellent. However, Lin's approach only restricts to the linear boundary conditions of phase change problems.

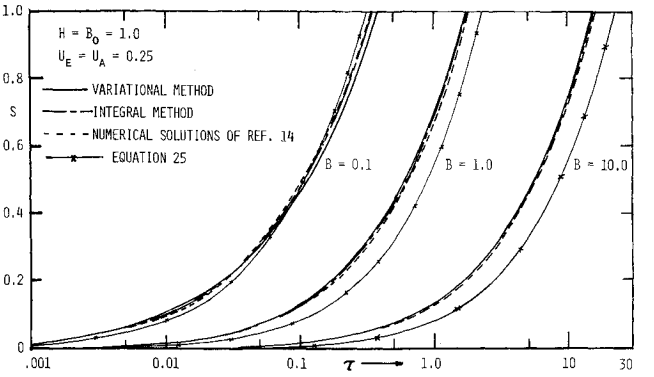


**Fig. 1 Comparisons between the present and the numerical solutions of Ref. 14 for the solidification front movement with time.**

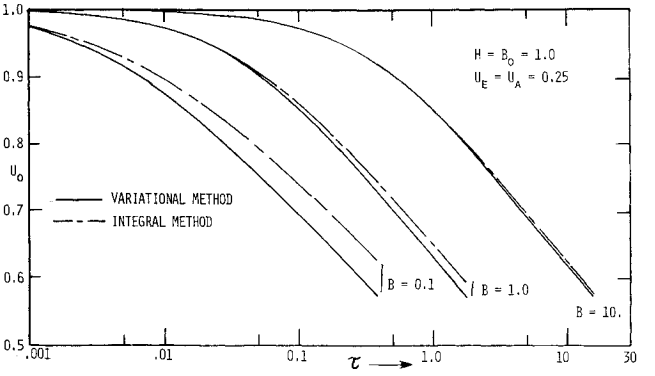


**Fig. 2 Temperature history at the radiation and convection surface.**

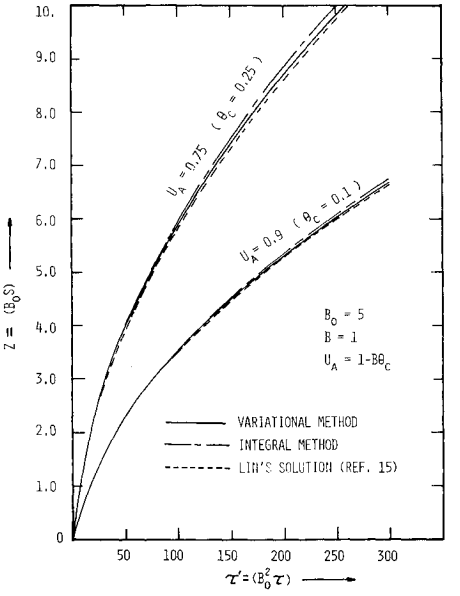


**Fig. 3 Comparisons between the present solutions with $H=0$ and Lin's solutions.**

It should be pointed out that the inclusion of parameter $B=10$ in Fig. 1 is strictly for comparison purposes. In practice, the value of $B$ for most metals and liquid is less than unity. Precisely speaking, the region of interest for $B$ is between 0.1 and 1. Table 1 summarizes the value of $B$ for a variety of material commonly met in engineering. The results are computed based on data of Ref. 17.

**Table 1 Values of $B$ for various materials**

| Material | $B$ |
|---|---|
| Aluminum | 0.390 |
| Aniline | 0.895 |
| Benzene | 0.815 |
| Beryllium | 0.513 |
| Bizmuth | 0.763 |
| Copper | 0.290 |
| Lead | 0.275 |
| Lithium | 0.090 |
| Magnesium | 0.308 |
| Mercury | 0.396 |
| Nickel | 0.306 |
| Sodium | 0.247 |
| Steel | 0.258 |
| Tin | 0.572 |
| Tungsten | 0.306 |
| Water | 0.293 |

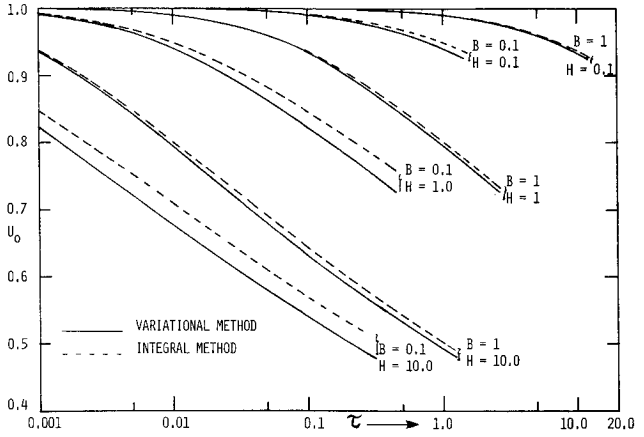


**Fig. 4 Temperature history at the radiative surface with $U_e = 0$ and $B_0 = 0$.**

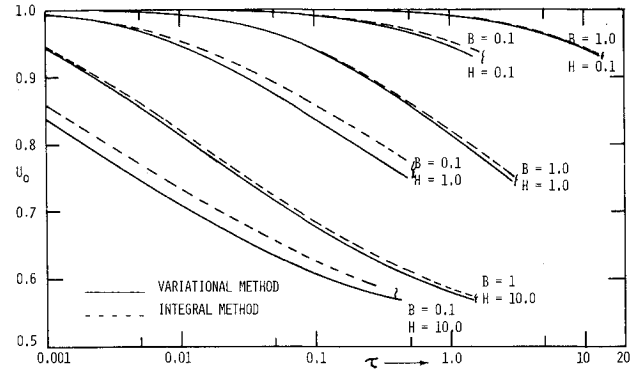


**Fig. 5 Temperature history at the radiative surface with $U_e = 0.25$ and $B_0 = 0$.**

Figures 4-7 display the surface temperature history of the newly formed solid phase subjected to radiative cooling only (i.e. $B_o = 0$). The temperature ratio between the environment and initial temperatures is taken as 0, 0.25, 0.5, and 0.75. In each figure the parameters of $B_o = 0.1$, 1.0, and $H = 0.1$, 1, and 10 are employed. It is found that the surface temperature drops faster as $T_e/T_i$ decreases which agrees with laws of physics. For a given value of $T_e/T_i$, the surface temperature drops slower as $B$ increases. In the figures, all the curves end at the total solidification time. As can be seen clearly, the total solidification time decreases as $B$, or $T_e/T_i$ decreases and increases as $H$ or $B_o$ decreases. However, the relations are not linear.

When $B = 0.1$ and $H = 10$ the surface temperature history predicted based on the variational method is somewhat lower than that of heat balance integral method. The maximum difference found in our computations is 5% with respect to the integral method. The discrepancy is partially due to the profile chosen in the solid. Present computations indicate that the coefficient $C$ in Eq. (15) is negligibly small if $B$ is in the order of magnitude of unity or higher. However, the second-order term in Eq. (15) becomes more significant when $B$ is less than 0.1. This finding further supports the statement made in the earlier analyses that the assumption of linear profile in the solid is adequate.[10, 11, 18]

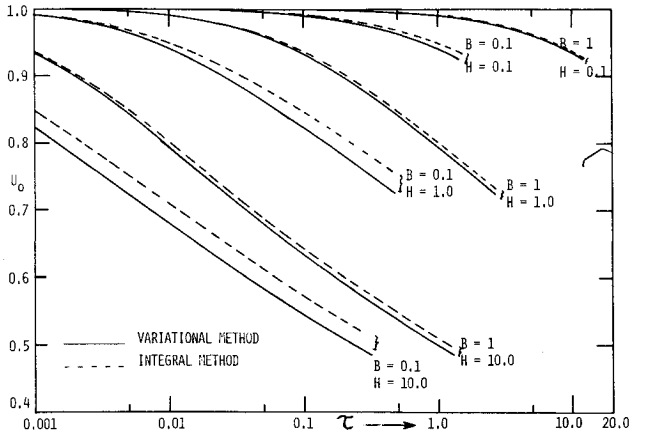


**Fig. 6 Temperature history at the radiative surface with $U_e = 0.25$ and $B_0 = 0$.**

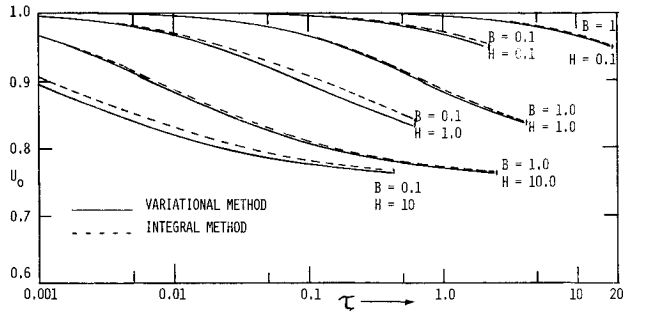


**Fig. 7 Temperature history at the radiative surface with $U_e = 0.75$ and $B_0 = 0$**

It should be pointed out that the second-order temperature profile

$$\theta = (q_I - T_i)(1 - X/\delta^2) \tag{23}$$

which has been successfully employed in the earlier analyses[19-21] in which the phase change is not involved, does not yield accurate results for the present problem.

The surface temperature and solidification rate obtained based on Biot's method with the temperature profile given by Eq. (23) are as follows

$$\dot{U}_o = f_2(U_o),\ U_o(0) = U_i \tag{24}$$

where

$$f_2(U_o) = F^3(U_o)[7(U_o - U_i)/30 - pB]/2 \div \{ [2B^2 - (2/3)\,pB(U_o - U_i) + (26/315)(U_o - U_i)^2]\,[F(U_o) - (U_o - U_i)(4HU_o^3 + B_o)] + (U_o - U_i)[(U_o - U_i)/21 - Bp/6]F(U_o)] \} \tag{24a}$$

$$S = [2(U_i - U_o)/F(U_o)] \tag{25}$$

Numerical examples based on Eqs. (24) and (25) are illustrated in Fig. 1. It appears that the aforementioned formulations overpredict the total solidification time (except for

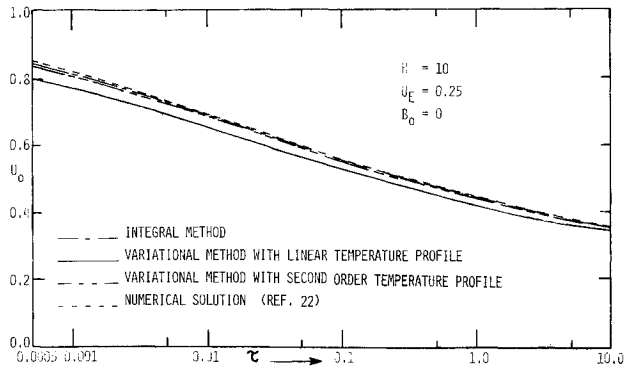


**Fig. 8 Comparisons between various approximate solutions with $B=0$ and the numerical solutions of Ref. 22.**

$B<0.1$). Evidently, the temperature profile given by Eq. (15) has the advantage of flexibility over that given by Eq. (8) or Eq. (23).

Under the el condition of $B=0$, $dS/d\tau$ is meaningful only when $\tau=0_+$, since all liquid or solid already completely changes to a new phase at $\tau>0$. Therefore, Eq. (21) yields the limit

$$\lim_{\tau\to 0} dS/d\tau \to \infty \text{ since } \lim_{\tau\to 0} S\to 0.$$

It is seen that the first limit agree with our intuition. However, when $B=0$, and $\tau>0$, $S$ no longer maintains the physical meaning of solidification or melt distance. Under this condition, $S$ is interpreted as "penetration depth" of the material without phase change. This can be easily seen from Eqs. (1-5). Note that Eq. (5) now reduces to $\partial T/\partial x=0$ at $x=\delta$.

When there is no phase change it is quite accurate to assume a parabolic temperature profile in the solid. This can be demonstrated by setting $B$ equal to zero in Eqs. (14, 22, and 24). These limiting solutions which lead to the surface temperature history in a semi-infinite solid subjected to radiative cooling at the surface are graphically shown in Fig. 8. Solutions based on a finite difference scheme reproduced from an earlier analysis[22] are also included. As can be seen, the agreement among all the solutions is good. However, the solid line which represents the results of variational method associated with the linear temperature profile is somewhat lower than the numerical solution. The maximum difference is found to be 9%.

It should be pointed out that the present analysis is restricted to one-dimensional heat conduction only. The model can be equally applied to the solidification of a semi-infinite region of fluid (the "total solidification time" becomes meaningless for this case). Similar techniques can be used for one-dimensional heat transfer in sphere and cylinder with phase change.Analysis involving temperature dependent properties is underway.[23]

## Conclusion

Problems of one-dimensional heat transfer involving solidification and melting of materials subject to radiation and convection are studied. Two different approximate analytical approaches are employed, based on which the complicated nonlinear governing equations are reduced to a simple initial value problem which can be easily integrated numerically with the available standard techniques. It is found that the total solidification time and the rate of solidifica obtained from the present analyses agree very well with the numerical solutions of Ref. 14. However, the present analyses are much simpler and more straightforward as compared with the numerical solutions directly using the finite difference sheme in which the additional problems of stability and convergence of solutions are involved in the computations.

## References

[1] Carslaw, H.S. and Jaeger, J.C., *Conduction of Heat in Solids* 2nd ed., Clarendon Press, Oxford, 1959.

[2] Stefan, J., "Uber die Theorie der Einsbildung Insbesonder uber die Eisbilding in Polarmaere," *Akad. Mat. Natur.*, Vol. 98, 1889 pp. 965-983.

[3] Biot, M.A., "New Methods in Heat Flow Analysis with Application to Flight Structures," *Journal of the Aeronautical Sciences,* Vol. 24, Dec. 1957, pp. 857-873.

[4] Biot, M.A., "Further Developments of New Methods in Heat-Flow Analysis," *Journal of the Aerospace Sciences,* Vol. 26, June 1959, pp. 367-381.

[5] Biot, M.A., *Variational Principle in Heat Transfer,* Clarendon Press, Oxford, 1970.

[6] Goodman, T.R., "The Heat-Balance Integral and Its Application to Problems Involving a Change of Phase," *Transactions of the ASME,* Vol. 80, Feb. 1958, pp. 335-342.

[7] Biot, M.A. and Daughaday, H., "Variational Analysis of Ablation," *Journal of the Aerospace Sciences,* Vol. 29, Feb. 1962, pp. 227-229.

[8] Biot, M.A. and Agrawal, H.C., "Variational Analysis of Ablation for Variable Properties," *Journal of Heat Transfer, Transactions of the ASME,* Vol. 86, Aug. 1964, pp. 437-442.

[9] Lardner, T.J., "Approximate Solutions to Phase-Change Problems," *AIAA Journal,* Vol. 5, Nov. 1967, pp. 2079-2080.

[10] Prasad, A. and Agrawal, H.C., "Biot's Variational Principle for Aerodynamic Ablation of Melting Solids," *AIAA Journal,* Vol. 12, Feb. 1974, pp. 250-252.

[11] Prasad, A. and Agrawal, "Biot's Variational Principle for a stefan Problem," *AIAA Journal,* Vol. 10, March 1972, pp. 325-327.

[12] Imber, M. and Huang, P.N.S., "Phase Change in a Semi-Infinite Solid with Temperature Dependent Thermal Properties," *International Journal of Heat and Mass Transfer,* Vol. 16, Oct. 1973, pp. 1951-1954.

[13] Goodling, J.S. and Khader, M.S., "Inward Solidification with Radiation-Convection Boundary Condition," *Journal of Heat Transfer, Transactions of ASME,* Vol. 96, Feb. 1974, pp. 114-115.

[14] Goodling, J.S. and Khader, M.S., "One-Dimensional Solidification with a Radiative Boundary," *Proceedings of the 10th Southeastern Seminar on Thermal Sciences,* 1974, pp. 286-303.

[15] Rosen, S.S., "The Temperature Distribution in a Semi-infinite Solid with Variable Thermo-physical Properties Subjected to Convection and/or Radiation at the Boundary," Ph.D. thesis, 1970, Department of Mechanical Engineering, Polytechnic Institute of Brooklyn, New York.

[16] Sui, L., "An Analytical Method for Solving Geometric One-Dimensional Freezing or Melting Problems," ASME Paper 73-WA/HT-33, 1973.

[17] Kreith, F. and Romie, F.E., "A Study of the Thermal Diffusion Equation with Boundary Conditions Corresponding to Solidification or Melting of Materials Initially at the Fusion Temperature," *Proceedings of the Physics Society,* Vol. 68, 1955, pp. 277-291.

[18] Hrycak, P., "Problem of Solidification with Newton's Cooling at Surface," *AIChE Journal,* Vol. 9, Sept. 1963, pp. 585-589.

[19] Lardner, T.J., "Biot's Variational Principle in Heat Conduction," *AIAA Journal,* Vol. 1, Jan. 1963, pp. 196-206.

[20] Zyszkowski, W., "The Transient Temperature Distribution in One-Dimensional Heat Conduction Problems with Nonlinear Boundary Conditions," *Journal of Heat Transfer, Transactions of the ASME,* Vol. 91, Feb. 1969, pp. 77-82.

[21] Chung, B.T.F. and Yeh, L.T., "Temperature Distribution in Slabs with Non-Uniform Heat Generations and Radiative Boundary Conditions," *24th Canadian Chemical Engineering Conference,* Ottawa, Canada, 1974.

[22] Chung, B.T.F. and Yeh, L.T., "Integral Method for Nonlinear Transient Heat Transfer in a Semi-infinite Solid," *Journal of Spacecraft and Rockets,* Vol. 10, Jan. 1973, pp. 88-89.

[23] Yeh, L.T.., "Biot's Variational Principle to Some Transient Heat Transfer Process Involving Thermal Radiation," Ph.D. thesis, Department of Mechanical Engineering. University of Akron, Ohio (in progress.)